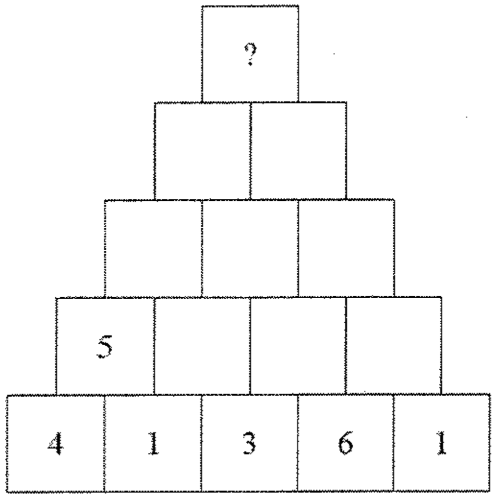
2012 John O'Bryan Mathematical Competition
5-person Team Test

Abbreviated Instructions: Answer each of the following questions **using separate sheet(s) of paper for each numbered problem**. Place your team letter in the upper right corner of each page that will be turned in (failure to do this will result in no score). Place problem numbers in the upper left corner. Problems are equally weighted; **<u>teams must show complete solutions (not just answers) to receive credit</u>**. More specific instructions are read verbally at the beginning of the test.

1. An N-row "Pascal-like" triangle is formed by filling the bottom row with numbers and then calculating the value of remaining entries by adding together the two entries that are in the two squares directly below the entry to be calculated. For example, the 5 found in the diagram comes from 4 + 1.



   a. Complete the diagram at the right and give the value of the top entry.

   b. In general, if we fill the fifth row with numbers *a, b, c, d,* and *e,* what will the value of the top entry be?

   c. Prove your result from part (b).

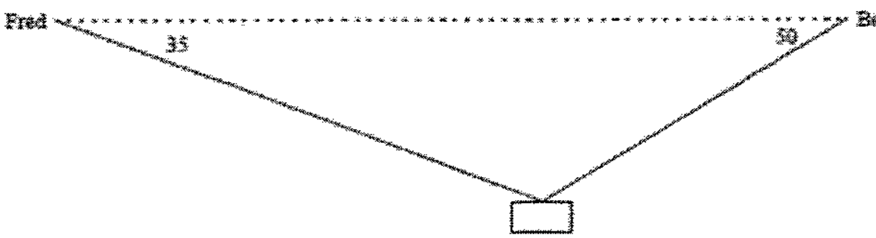
   d. If we have a triangle having N rows, how would one calculate the top entry <u>without calculating any of the intermediate entries</u> between rows one and N?

2. In a word-guessing game, the goal is to guess a chosen word of known length. With each guess, the guesser is told whether the guessed word matched the target word in an even or odd number of positions (to match, letters must be in the same spot as in the target word, so for example ATE matches only once to THE; zero is counted as even). For example, if the target word is WATER and the guess is WAVER, the reply would be "even" since four places have been matched. If six rounds have been played with the guesses and replies shown in the box at the right, what is the target word?

Guess	**Result**
DAY	even
SAY	odd
MAY	even
DUE	odd
BUY	even
TEN	odd

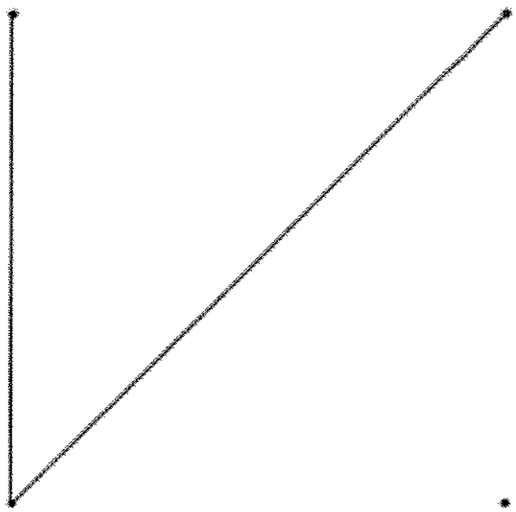
3. Inscribe a rectangle of base b and height h in a circle of radius 1, and inscribe an isosceles triangle in a region of the circle cut off by one base of the rectangle (with that base becoming also one side of the triangle). For what value(s) of h can one construct these shapes as described such that the rectangle and triangle have the same area?

4. A weight hangs on a rope of length 100 feet as shown in the diagram below. The rope hangs with an angle of 50 degrees from the horizontal on Beth's end and 35 degrees from the horizontal on Fred's end. How far below the level of Beth and Fred is the weight?

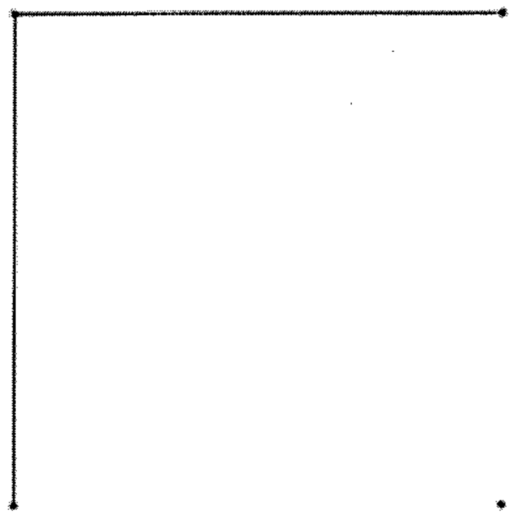


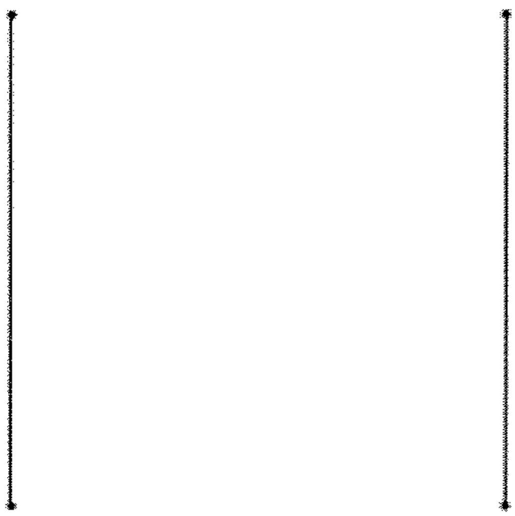
5. The digits of 2012 sum to 5. Since the year one, in how many years have the digits of the year summed to five? (Remember to fully justify your answer!)

6. Given four vertices, a "graph" is created by joining any number of the vertices by straight line(s). Two graphs are considered **equivalent** if it is possible to rearrange the vertices in such a way as to achieve the same picture. Two examples of equivalent graphs are given below:

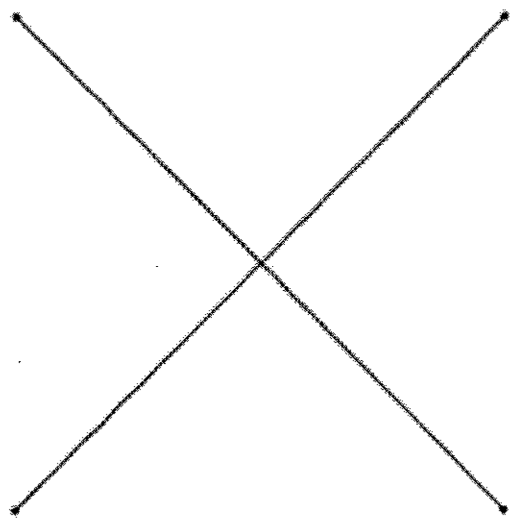


is equivalent to





is equivalent to



Find all non-equivalent graphs of four vertices. Note that points will be deducted for the inclusion of equivalent graphs in your final answer.

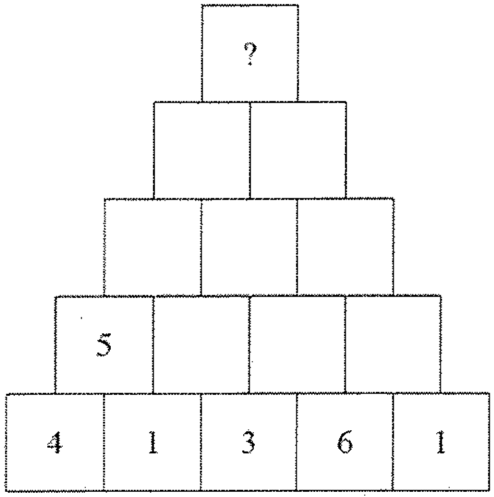
Solutions to these problems will be posted after the contest to:

http://www.nku.edu/~math/job/

2012 John O'Bryan Mathematical Competition
5-person Team Test - Solutions

Problems are equally weighted; **teams must show complete solutions (not just answers) to receive credit.**

1. An N-row "Pascal-like" triangle is formed by filling the bottom row with numbers and then calculating the value of remaining entries by adding together the two entries that are in the two squares directly below the entry to be calculated. For example, the 5 found in the diagram comes from 4 + 1.



   a. Complete the diagram at the right and give the value of the top entry.

   Fourth row: 5 4 9 7
   Third Row: 9 13 16
   Second Row: 22 29
   Top Row: 51

   b. In general, if we fill the fifth row with numbers *a, b, c, d,* and *e,* what will the value of the top entry be?

   1(a)+4(b)+6(c)+4(d)+1(e). In part (a) this is 4+4(1)+6(3)+4(6)+1 = 51. The numbers 1 4 6 4 1 are the entries of the fourth (start counting with zero) row of Pascal's triangle: C(4,0), C(4,1), etc.

   c. Prove your result from part (b).

   The fourth row is (a+b) (b+c) (c+d) (d+e).
   The third is (a+2b+c) (b+2c+d) (c+2d+e).
   The second is (a+3b+3c+d) (b+3c+3d+e).
   So the top entry is (a+4b+6c+4d+e).

   Since all the coefficients in each row are numbers in Pascal's triangle, we could also use properties of these numbers to prove the result.

   d. If we have a triangle having N rows, how would one calculate the top entry <u>without calculating any of the intermediate entries</u> between rows one and N?

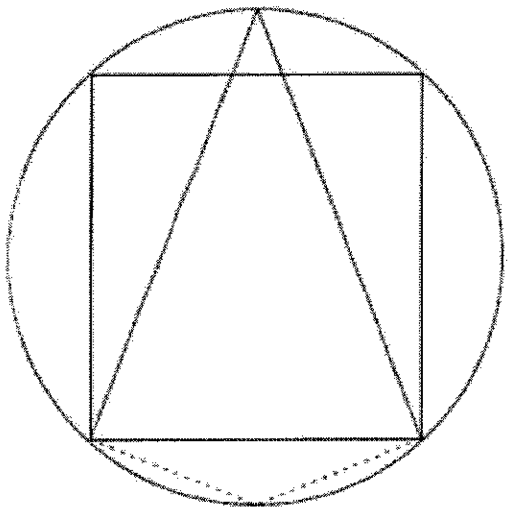
   Multiply the entries of the nth row (or n+1st row depending on how you number them) of Pascal's triangle by the entries in the bottom row. So if the bottom row were 1 2 3 4 5 6, the top number would be 1+5(2)+10(3)+10(4)+5(5)+6 = 112.

2. In a word-guessing game, the goal is to guess a chosen word of known length. With each guess, the guesser is told whether the guessed word matched the target word in an even or odd number of positions (to match, letters must be in the same spot as in the target word, so for example ATE matches only once to THE; zero is counted as even). For example, if the target word is WATER and the guess is WAVER, the reply would be "even" since four places have been matched. If six rounds have been played with the guesses and replies shown in the box at the right, what is the target word?

Guess	Result
DAY	even
SAY	odd
MAY	even
DUE	odd
BUY	even
TEN	odd

For DAY and MAY to be even either the AY matched, or nothing matched. If the AY matched, then the BUY says that B_Y matches and the word is BAY. But then SAY is not Odd. So niether the A nor the Y match. So we get S must match from SAY. From the other two Odds we get either SEE or SUN. But BUY is even and so only SEE is left as a possibility.

3. Inscribe a rectangle of base b and height h in a circle of radius 1, and inscribe an isosceles triangle in a region of the circle cut off by one base of the rectangle (with that base becoming also one side of the triangle). For what value(s) of h can one construct these shapes as described such that the rectangle and triangle have the same area?



This can be interpreted in two ways, either as the solid triangle shown in the picture or as the triangle formed from the two dotted lines and the base of the rectangle.

Let $\mathbf{b}$ *be the base of the rectangle and* $\mathbf{h}$ *be its height.*

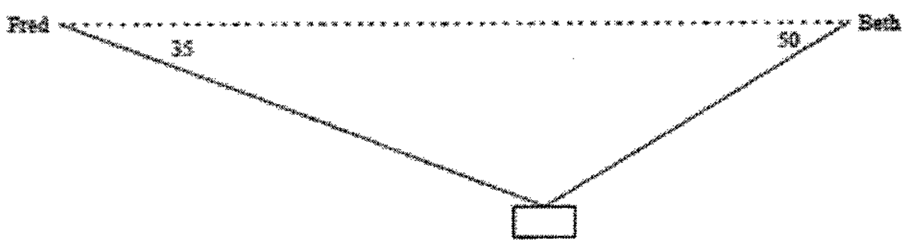
Using the triangle including the dotted lines, since the diameter is 2, the height of the triangle is $\frac{1}{2}(2-\mathbf{h}) = 1-\mathbf{h}/2$. *Thus the area of the triangle is* $\frac{1}{2}(\mathbf{b})(1-\mathbf{h}/2)$. *Setting this equal to the area of the rectangle* $\mathbf{bh}$, *we find* $\mathbf{h} = 2/5$.

Using the triangle including the solid lines, the height will be $1+\mathbf{h}/2$ *since the radius is one. Thus the area of the solid triangle is* $\frac{1}{2}(\mathbf{b})(1+\mathbf{h}/2)$. *Setting this equal to* $\mathbf{bh}$ *yields* $\mathbf{h} = 2/3$.

The possible values of H are 2/3 and 2/5.

4. A weight hangs on a rope of length 100 feet as shown in the diagram below. The rope hangs with an angle of 50 degrees from the horizontal on Beth's end and 35 degrees from the horizontal on Fred's end. How far below the level of Beth and Fred is the weight?

Let $\mathbf{h}$ *be the distance we are looking for. Let* $\mathbf{x}$ *be the length of rope from Fred to the weight, so that* $100-\mathbf{x}$ *is the length of rope from Beth to the weight. Then we have:*



$$\sin(35) = \mathbf{h}/\mathbf{x}$$
$$\sin(50) = \mathbf{h}/(100-\mathbf{x})$$

Solving by substitution ($\mathbf{h} = \mathbf{x}\sin(35)$) *yields* $\mathbf{x} = (100 \sin 50)/(\sin 50 + \sin 35) = 57.1837$.
Then $\mathbf{h} = 57.1837 \sin 35 = 32.7992$ *feet.*

5. The digits of 2012 sum to 5. Since the year one, in how many years have the digits of the year summed to five? (Remember to fully justify your answer!)

Assume a four-digit year. If the thousands digit is zero, then the other three digits sum to five. This means they are 500, 410, 320, 311, or 221. There are three ways to arrange 500, 311, and 221 and six ways to arrange 410 and 320. Thus between the years 1 and 999 there are 21 such years.

If the thousands digit is 1, then the options become 400, 310, 211, 220 for a total of 3+6+3+3 = 15 ways.

In the 2000's, only 2003 and 2012 sum to 5. Thus there are 21 + 15 + 2 = 27 years between year one and year 2012 in which the digits summed to five.

6. Given four vertices, a "graph" is created by joining any number of the vertices by straight line(s). Two graphs are considered **equivalent** if it is possible to rearrange the vertices in such a way as to achieve the same picture. Find all non-equivalent graphs of four vertices. Note that points will be deducted for the inclusion of equivalent graphs in your final answer.